Kord Technologies

Conservation Law Based Updating Schemes for Overset Meshes

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Conservation Law Based Update Schemes

• In this talk a method for the exchange of data at overset mesh interfaces is outlined

• The method is based on enforcing the conservation laws on a cell-wise basis rather than the traditional approaches involving interpolation

• Reexamine some old questions in a new light

• Consider the following conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

Conservation Law Based Update Schemes

- In order to solve this equation numerically on a single grid, the domain is partitioned into discrete cells $j = \left[x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2} \right]$

- The variable q is averaged over cell j as follows

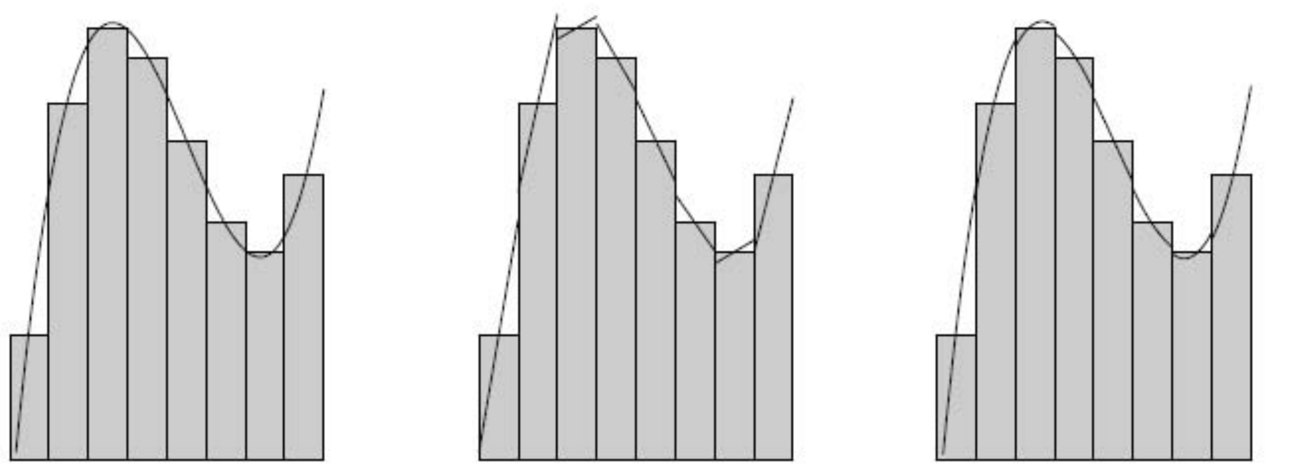
$$q_j^n = \int_{x_j - \frac{\Delta x}{2}}^{x_j + \frac{\Delta x}{2}} q(x, t_n) dx \ ; t_n = \mathrm{n}\, \Delta \mathrm{t} \, ; \mathrm{etc.}$$

- The value of the cell averaged variable q_j in cell j is updated as follows

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\Delta x} \left(F\left(ql_{j+1}, qr_{j+1}\right) - F\left(ql_j, qr_j\right) \right)$$

Conservation Law Based Update Schemes

- Here ql, qr are the values of q reconstructed from the cell averages q_j^n and evaluated at the cell interfaces using the MUSCL procedure and F is a numerical flux function (Roe, HLLE++, etc.)



a. Cell averaging of quartic data b. Linear reconstruction c. Quadratic reconstruction **(Barth and Ohlberger, 2004)**

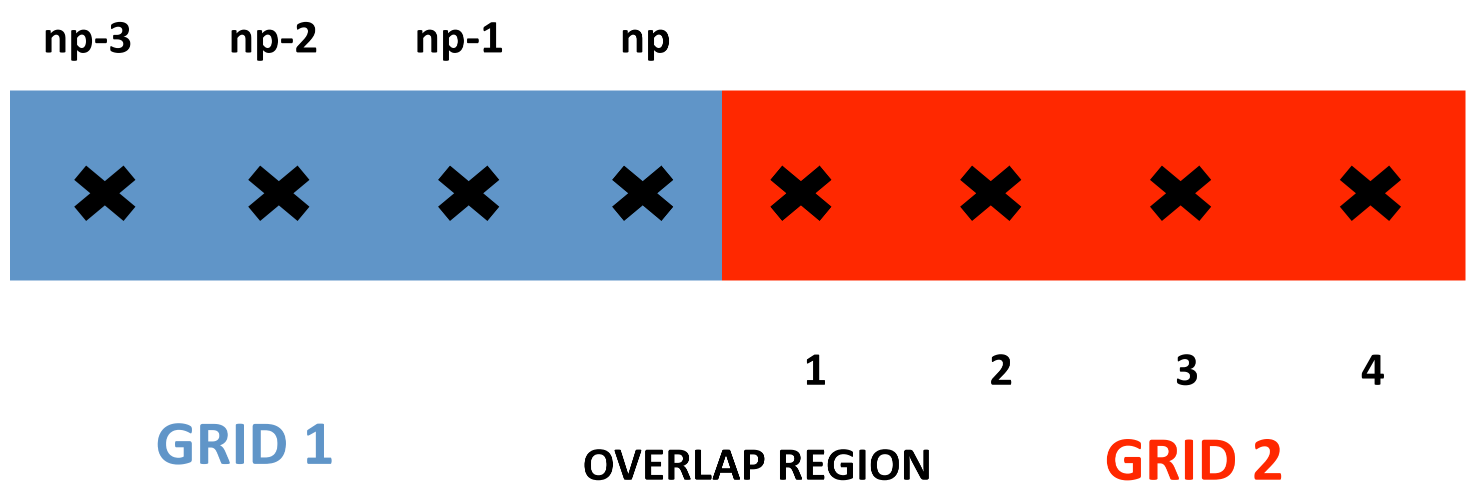
Conservation Law Based Update Schemes

• What is the "best" way to extend this approach to cases of multiple overlapping grids?

•Rather than interpolate variables or fluxes seek alternate formulation

•Use data from both zones to reconstruct interface states

•Consider two grids with some small amount of overlap

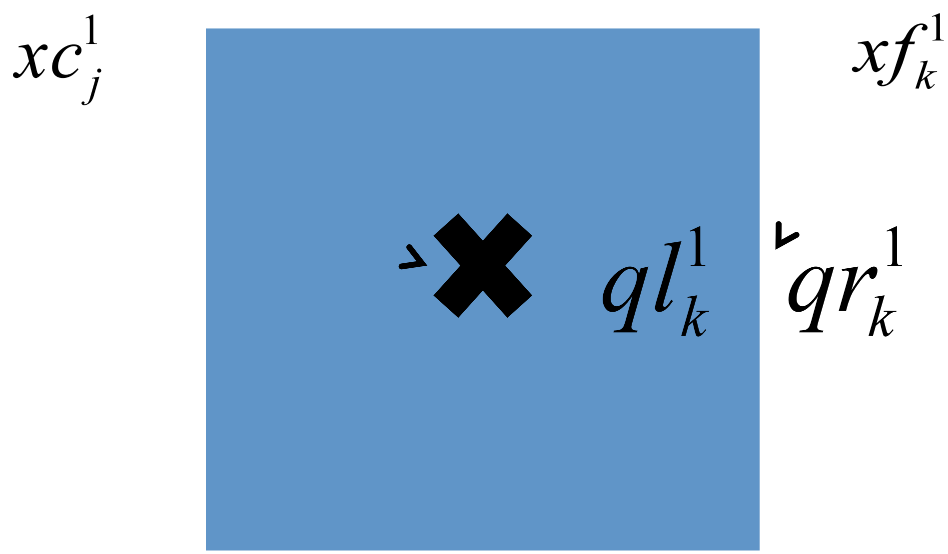


Conservation Law Based Update Schemes

•Let the value of the cell averaged variable in cell *j* of grid 1 be denoted as q^1_j, and the cell centroid value of cell j be denoted as xc^1_j etc

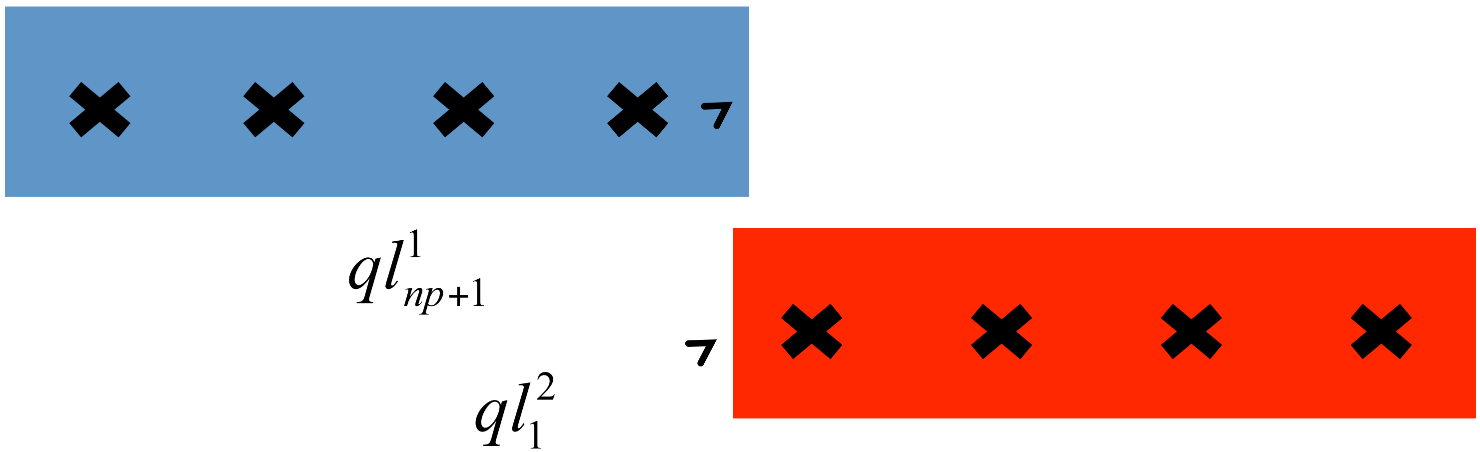
•Let the left and right interface states at face *k* of grid 1 be denoted as ql^1_k and qr^1_k respectively

•Let the face centroid of face *k* of grid 1 be denoted as xf^1_k respectively

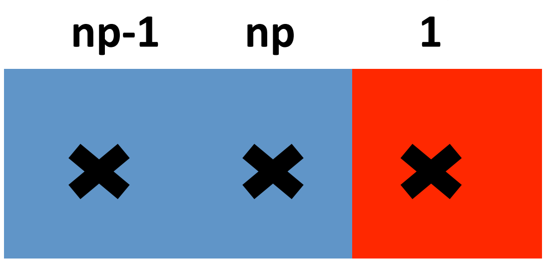


Conservation Law Based Update Schemes

- Consider computing the cell interface states ql^1_{np+1} and ql^2_1 as follows



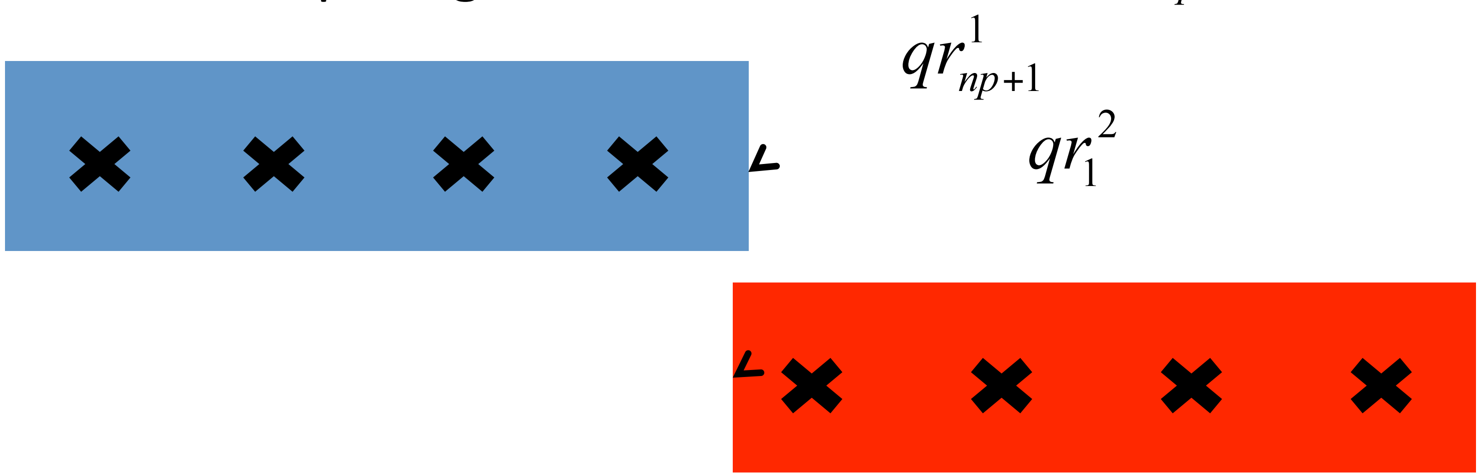
- The states are computed by constructing a least squares monotonically limited approximation to the solution gradient in cell np from mesh 1 using the cells below and then extrapolating q to the cell faces using this gradient



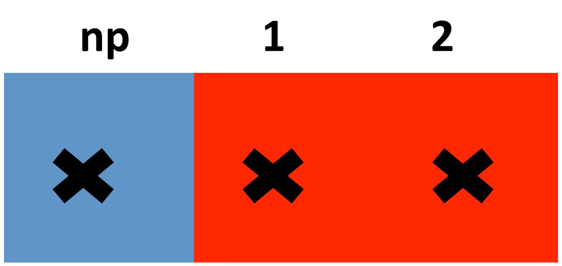
$$
\begin{aligned}
ql^1_{np+1} &= q^1_{np} + \overline{\nabla} q^1_{np} \bullet \left(xf^1_{np+1} - xc^1_{np} \right) \\
ql^2_1 &= q^1_{np} + \overline{\nabla} q^1_{np} \bullet \left(xf^2_1 - xc^1_{np} \right)
\end{aligned}
$$

Conservation Law Based Update Schemes

- Consider computing the cell interface states qr^1_{np+1} and qr^2_1 as follows



- The states are computed by constructing a least squares monotonically limited approximation to the solution gradient in cell 1 from mesh 2 using the cells below and then extrapolating q to the cell faces using this gradient



$$
\begin{aligned}
qr^1_{np+1} &= q^2_1 + \overline{\nabla} q^2_1 \bullet \left(xf^1_{np+1} - xc^2_1\right) \\
qr^2_1 &= q^2_1 + \overline{\nabla} q^2_1 \bullet \left(xf^2_1 - xc^2_1\right)
\end{aligned}
$$

Conservation Law Based Update Schemes

•A consistent and convergent scheme should be achieved as grid refinement is performed if the distance between the cells centers used in the reconstructions approach zero under grid refinement, i.e.

$$ql^{1}_{np+1} \rightarrow ql^{2}_{1} \text{ and } qr^{1}_{np+1} \rightarrow qr^{2}_{1} \text{ as } \Delta x \rightarrow 0$$

in that

$$\left\| qr^{1}_{np+1} - qr^{2}_{1} \right\| = \left\| \overline{\nabla} q^{2}_{1} \bullet \left(xf^{1}_{np+1} - xf^{2}_{1} \right) \right\| \leq \left\| \overline{\nabla} q^{2}_{1} \right\| \left\| \left(xf^{1}_{np+1} - xf^{2}_{1} \right) \right\| \leq TV(q)\Delta x$$

etc.

•Here TV is the total variation

•A similar argument holds for meshes with small gaps between the grids

•The basic argument remains unchanged in the case of general overlap

Generalized Lax-Wendroff Theorem

- Recall that a function $q(x,t)$ is considered to be a weak solution to the conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

If for any compact, differentiable function $\phi(x,t)$ the following integrals are satisfied

$$\int_0^{\infty} \int_{-\infty}^{\infty} \left[\phi_t q + \phi_x f(q)\right] dx\, dt = -\int_{-\infty}^{\infty} \phi(x,0) q(x,0)\, dx$$

Generalized Lax-Wendroff Theorem

- In order to demonstrate that these integrals are satisfied for the proposed method, first multiply the cell average update equation by $\phi\left(x_j^1, t_n\right)$ for mesh 1, etc.

$$\phi(x_j^1,t_n)\left(q_j^{1,n+1} - q_j^{1,n}\right) = -\frac{\Delta t}{\Delta x}\phi(x_j^1,t_n)\left(F\left(ql_{j+1}^1, qr_{j+1}^1\right) - F\left(ql_j^1, qr_j^1\right)\right)$$

- Next sum over j and n for each mesh as follows

$$\Delta t \Delta x \left\{ \sum_{j=-\infty}^{np} \sum_{n=0}^{\infty} \phi(x_j^1,t_n)\frac{\left(q_j^{1,n+1} - q_j^{1,n}\right)}{\Delta t} + \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \phi(x_j^2,t_n)\frac{\left(q_j^{2,n+1} - q_j^{2,n}\right)}{\Delta t} = \right.$$

$$- \sum_{j=-\infty}^{np} \sum_{n=0}^{\infty} \phi(x_j^1,t_n)\frac{\left(F\left(ql_{j+1}^1, qr_{j+1}^1\right) - F\left(ql_j^1, qr_j^1\right)\right)}{\Delta x}$$

$$\left. - \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \phi(x_j^2,t_n)\frac{\left(F\left(ql_{j+1}^2, qr_{j+1}^2\right) - F\left(ql_j^2, qr_j^2\right)\right)}{\Delta x} \right\}$$

Generalized Lax-Wendroff Theorem

- The following summation by parts formulas are useful for the proving convergence of the proposed scheme to a weak solution

$$\sum_{i=0}^{N} a_j\left(b_j - b_{j-1}\right) = a_N b_N - a_0 b_{-1} - \sum_{i=0}^{N-1} b_j\left(a_{j+1} - a_j\right) \quad \text{SPB1}$$

$$\sum_{i=0}^{N} a_j\left(b_{j+1} - b_j\right) = a_N b_{N+1} - a_0 b_0 - \sum_{i=1}^{N} b_j\left(a_j - a_{j-1}\right) \quad \text{SPB2}$$

Generalized Lax-Wendroff Theorem

- For the LHS use the SPB2 formula on the n sum

$$-\sum_{j=-\infty}^{np}\Delta x\,\phi(x_j^1,t_0)q_j^{1,0} - \sum_{j=1}^{\infty}\Delta x\,\phi(x_j^2,t_0)q_j^{2,0}$$

$$-\sum_{n=1}^{\infty}\Delta t\left(\sum_{j=-\infty}^{np}\Delta x\,q_j^{1,n}\left(\frac{\phi(x_j^1,t_n)-\phi(x_j^1,t_{n-1})}{\Delta t}\right)+\sum_{j=1}^{\infty}\Delta x\,q_j^{2,n}\left(\frac{\phi(x_j^2,t_n)-\phi(x_j^2,t_{n-1})}{\Delta t}\right)\right)$$

- Note that these terms are discrete approximations to the following integrals

$$-\int_{-\infty}^{\mathrm{xf}_{\mathrm{np+1}}^{1}}\phi(x,0)q(x,0)dx-\int_{\mathrm{xf}_1^2}^{\infty}\phi(x,0)q(x,0)dx-\int_0^{\infty}\left(\int_{-\infty}^{\mathrm{xf}_{\mathrm{np+1}}^{1}}\phi_t q+\int_{\mathrm{xf}_1^2}^{\infty}\phi_t q\right)dxdt$$

Generalized Lax-Wendroff Theorem

- For the RHS use the SPB1 formula on the j sum

$$\sum_{n=0}^{\infty} \Delta t \left(\sum_{j=-\infty}^{np} \Delta x \, F\left(ql_j^{\,1}, qr_j^{\,1}\right) \left(\frac{\phi(x_j^1, t_n) - \phi(x_{j-1}^1, t_n)}{\Delta x} \right) + \sum_{j=2}^{\infty} \Delta x \, F\left(ql_j^{\,2}, qr_j^{\,2}\right) \left(\frac{\phi(x_j^2, t_n) - \phi(x_{j-1}^2, t_n)}{\Delta x} \right) \right)$$

$$-\sum_{n=0}^{\infty} \Delta t \Delta x \left(F\left(ql_{np+1}^{1}, qr_{np+1}^{1}\right) \frac{\phi(x_{np}^1, t_n)}{\Delta x} - F\left(ql_1^{\,2}, qr_1^{\,2}\right) \frac{\phi(x_1^2, t_n)}{\Delta x} \right)$$

End Terms from SBP Formula

Generalized Lax-Wendroff Theorem

- The end terms at the grid interface become

$$-\sum_{n=0}^{\infty} \Delta t \Delta x \left(F\left(ql_{np+1}^{1}, qr_{np+1}^{1}\right) \frac{\phi(x_{np}^{1}, t_n)}{\Delta x} - F\left(ql_{1}^{2}, qr_{1}^{2}\right) \frac{\phi(x_{1}^{2}, t_n)}{\Delta x} \right) =$$

$$\sum_{n=0}^{\infty} \Delta t\, \Delta x \left(F\left(ql_{1}^{2}, qr_{1}^{2}\right) \left(\frac{\phi(x_{1}^{2}, t_n) - \phi(x_{np}^{1}, t_n)}{\Delta x} \right) + \frac{\Delta F_{12} \phi(x_{np}^{1}, t_n)}{\Delta x} \right)$$

where $\Delta F_{12} = F\left(ql_{1}^{2}, qr_{1}^{2}\right) - F\left(ql_{np+1}^{1}, qr_{np+1}^{1}\right)$

Generalized Lax-Wendroff Theorem

- We can show that $\Delta F_{12} \rightarrow 0$ as $\Delta x \rightarrow 0$ as follows

$$\Delta F_{12} = F\left(ql_1^{\ 2}, qr_1^{\ 2}\right) - F\left(ql_{np+1}^{\ 1}, qr_{np+1}^{\ 1}\right)$$

$$\Delta F_{12} = F\left(ql_1^{\ 2}, qr_1^{\ 2}\right) - F\left(ql_1^{\ 2} + (ql_{np+1}^{\ 1} - ql_1^{\ 2}), qr_1^{\ 2} + (qr_{np+1}^{\ 1} - qr_1^{\ 2})\right)$$

$$\left\|\Delta F_{12}\right\| \leq C \max\left(\left\|ql_{np+1}^{\ 1} - ql_1^{\ 2}\right\|, \left\|qr_{np+1}^{\ 1} - qr_1^{\ 2}\right\|\right)$$ **F is Lipschitz Continuous**

- Given that

$$\left\|qr_{np+1}^{\ 1} - qr_1^{\ 2}\right\| = \left\|\overline{\nabla} q_1^2 \bullet \left(xf_{np+1}^1 - xf_1^2\right)\right\| \leq \left\|\overline{\nabla} q_1^2\right\| \left\|\left(xf_{np+1}^1 - xf_1^2\right)\right\| \leq TV(q)\Delta x, \text{etc.}$$

it follows that $\Rightarrow 0 \text{ as } \Delta x \Rightarrow 0$

$$\left\|\Delta F_{12}\phi(x_{np}^1, t_n)\right\| \leq C\, TV(q^n)\Delta x \left\|\phi(x_{np}^1, t_n)\right\|$$

Generalized Lax-Wendroff Theorem

- Note that these terms are discrete approximations to the following integrals

$$\int_0^{\infty}\int_{-\infty}^{xf_{np+1}^1} f(q)\phi_x\,dx\,dt+\int_0^{\infty}\int_{xf_1^2}^{\infty} f(q)\phi_x\,dx\,dt$$

- If $\left\| xf_{np+1}^1 - xf_1^2 \right\| \to 0 \text{ as } \Delta x \to 0$ then terms like

$$-\int_{-\infty}^{\mathrm{xf}_{\mathrm{np+1}}^1} \phi(x,0)q(x,0)\,dx-\int_{\mathrm{xf}_1^2}^{\infty}\phi(x,0)q(x,0)\,dx-\int_0^{\infty}\left(\int_{-\infty}^{\mathrm{xf}_{\mathrm{np+1}}^1} \phi_t q+\int_{\mathrm{xf}_1^2}^{\infty}\phi_t q\right)dx\,dt$$

become $-\int_{-\infty}^{\infty}\phi(x,0)q(x,0)\,dx-\int_0^{\infty}\int_{-\infty}^{\infty}\phi_t q\,dx\,dt$

and the integral conservation law is satisfied under the same caveats invoked for the single grid case

Observations Based on Lax-Wendroff Theorem (1)

- If standard linear interpolation methods are used then no <u>GENERAL</u> bounds may be established a priori for the ΔF_{12} term

- We end up with a source term in the integral conservation law which MAY not vanish under grid refinement in the presence of discontinuous solutions.

$$\approx \int_0^{\infty} \Delta F_{12}(t)\,\varphi\left(x_R^1, t\right) dt$$

- These source terms become ODEs along characteristics and may globally pollute the solution
  - Corrupt convergence rates
  - Generate spurious waves

Observations Based on Lax-Wendroff Theorem (2)

- These terms can and will corrupt solutions when
  - Severe mismatches in cell sizes exist
  - Strong discontinuities exist in the grid interface region
  - Slow moving discontinuities exist in the grid interface region
  - ***All three exist simultaneously***!

- These facts need to be taken into consideration when designing grid "surgery" algorithms
  - helps explain the success of the "LEVEL2" interpolation approach used in PEGASUS5 and other grid connectivity codes

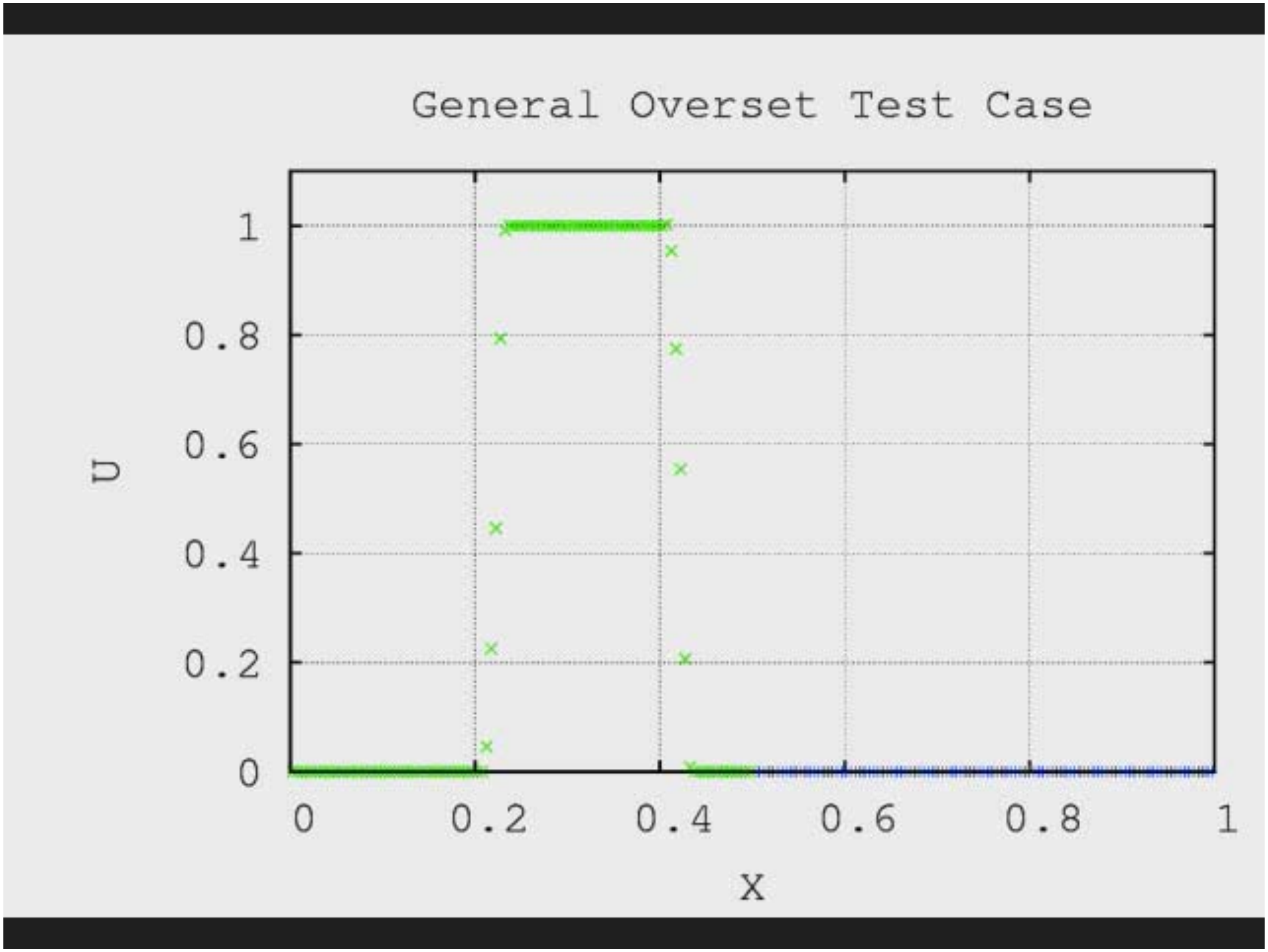
Examples Cases

- SIMPLE ADVECTION (LINEAR DISCONTINUITY)
- BURGERS EQUATION (NONLINEAR DISCONTINUITY)
- EULER EQUATION
  - SLOW MOVING SHOCK (use HLLE++ Riemann Solver)

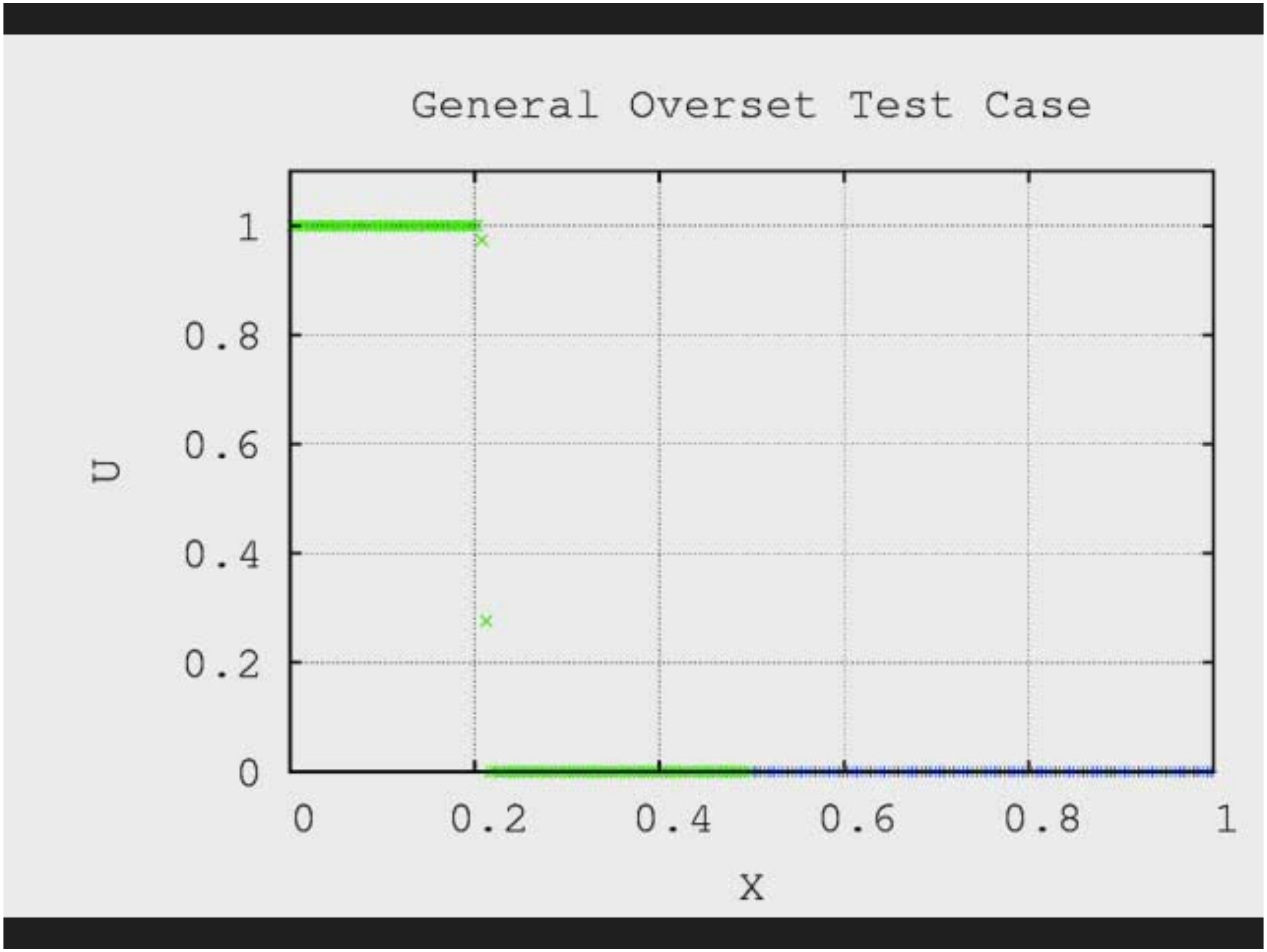
- All these cases have a small gap in between the grids!

- Use Barth-Jespersen Limiter

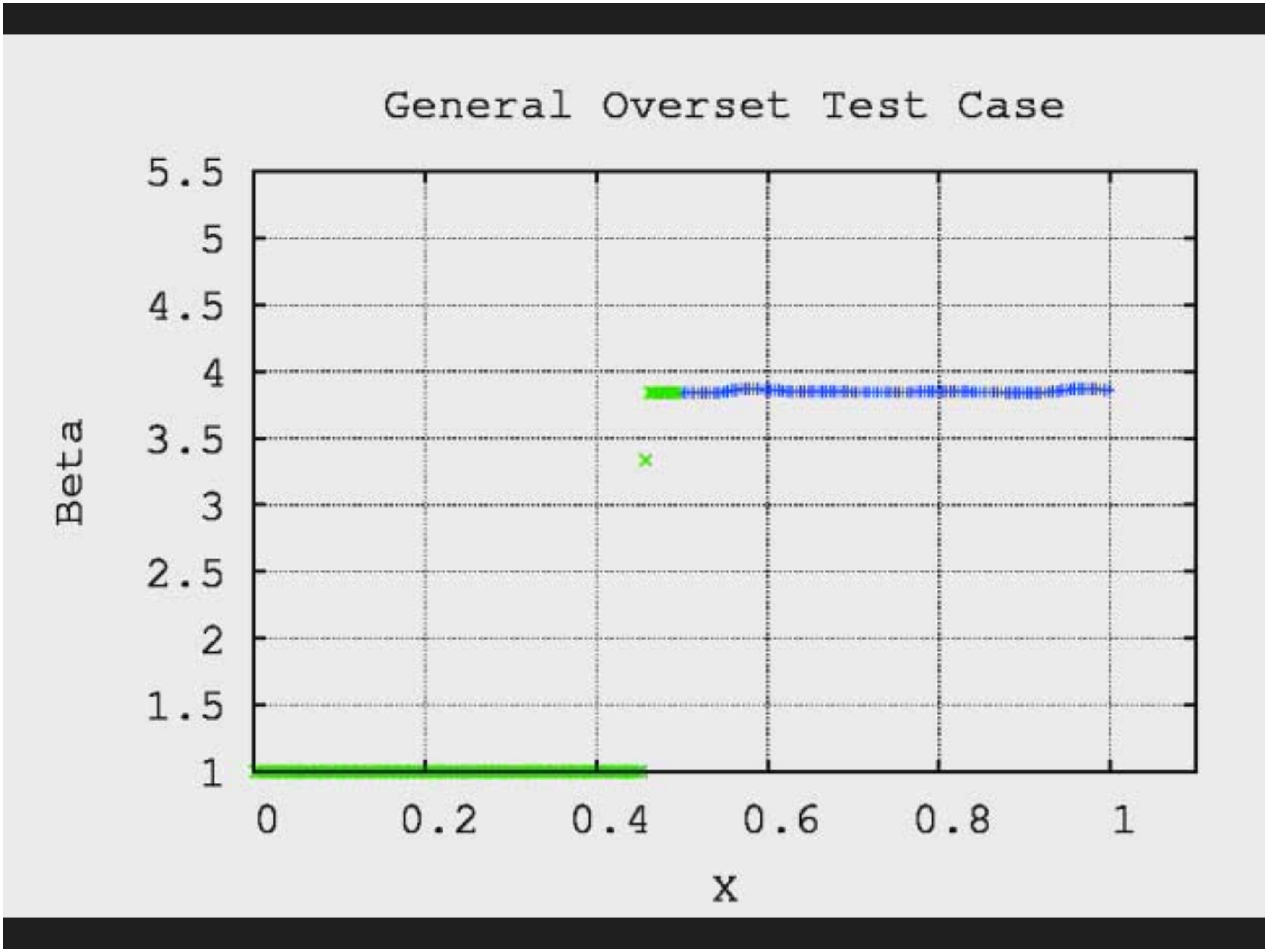
Simple Advection



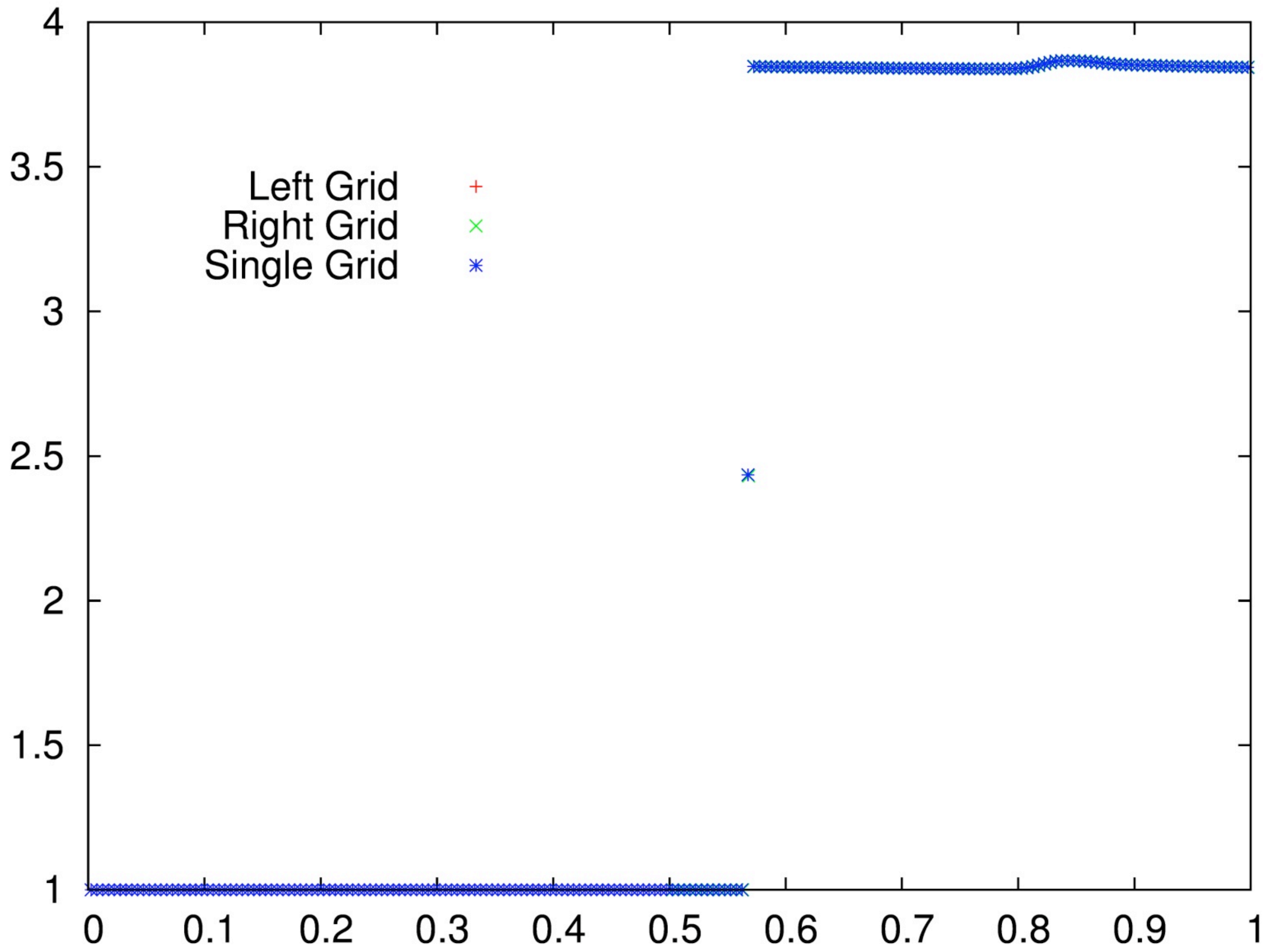
Burgers Equation



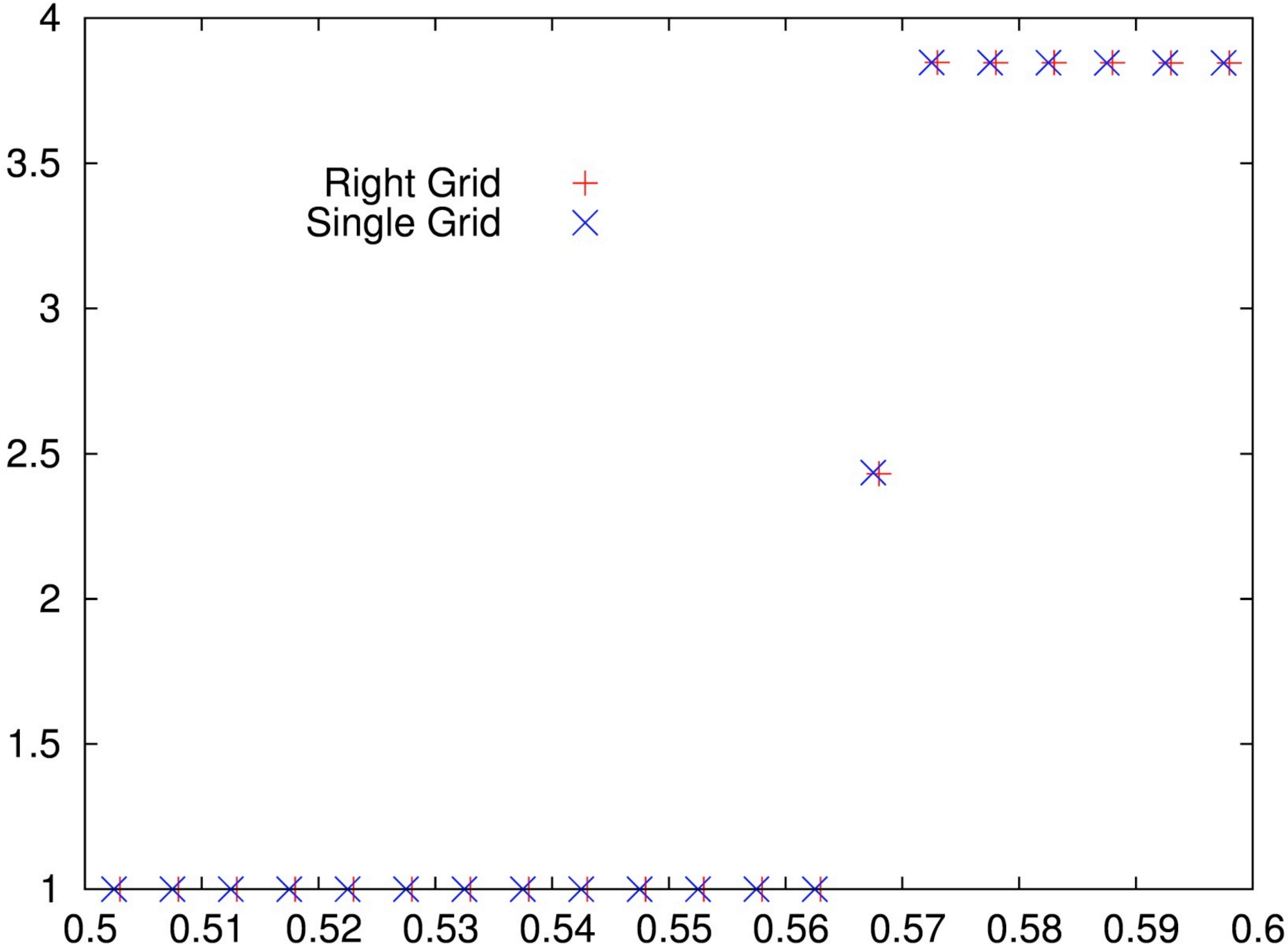
Slow Moving Shock (Euler)



Single Grid Comparison

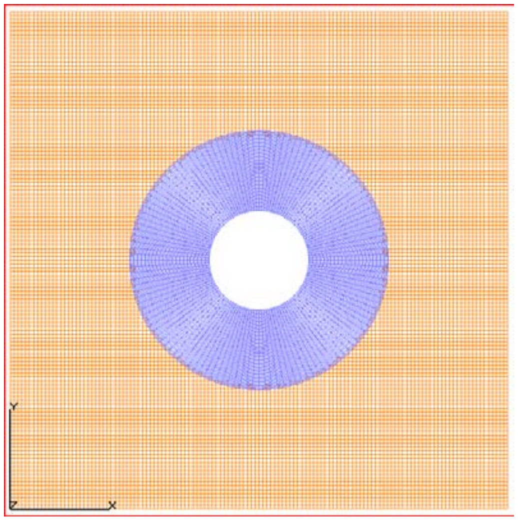


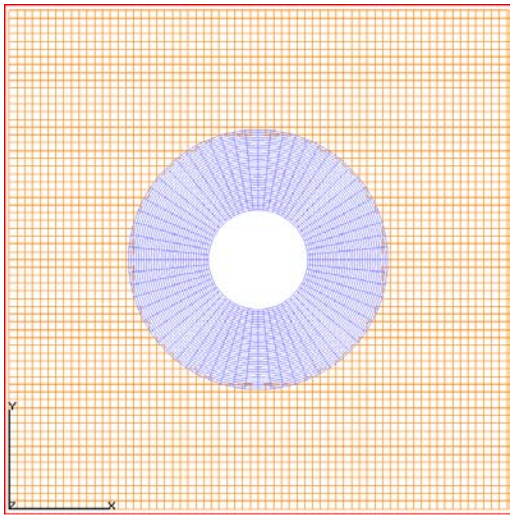
Single Grid Comparison

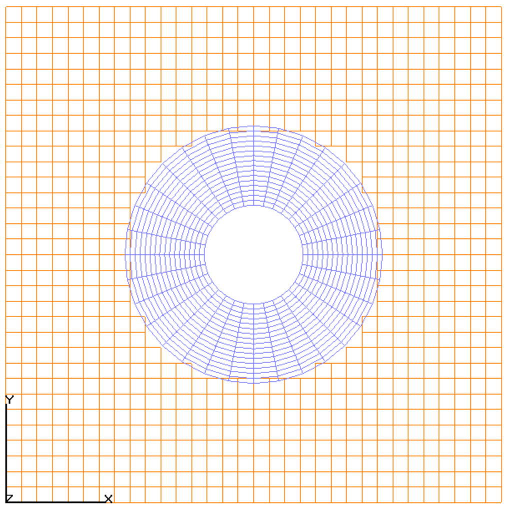


Conclusions

- Within the context of this framework strict lack of conservation becomes merely another source of numerical error to be removed by grid refinement
- Convergence to a weak solution is **guaranteed** as grid independence is achieved
- Method should be of some benefit to
  - Multigrid calculations
  - Grid convergence studies
    - No need to maintain large overlap in finest grids







Conclusions

- Need to develop algorithms for determining minimal overlap

- Achieving higher-order (> 3) accuracy problematic
  - Although a DG based chimera method appears to fit nicely within this framework

- The approach can be generalized to finite difference based codes by using a "finite-point" method at the interfaces

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That's All Folks

- Questions
- Constructive Criticism